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Research note

Study of Boundary Layer Convective Heat Transfer with Low Pressure Gradient Over a Flat Plate Via He's Homotopy Perturbation Method

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Abstract
The boundary layer convective heat transfer equations with low pressure gradient over a flat plate are solved using Homotopy Perturbation Method (HPM), which is one of the semi-exact methods. The nonlinear equations of momentum and energy solved simultaneously via HPM are in good agreement with results obtained from numerical methods. Using this method, a general equation in terms of Pr number and pressure gradient (λ) is derived which can be used to investigate velocity and temperature profiles in the boundary layer.

Keywords: *Homotopy Perturbation Method (HPM), Boundary Layer Convective Heat Transfer, Pressure Gradient*

1. Introduction
In many industrial applications, the problems related to forced convection in large pipes or on the surface of the turbomachine-blades can be reduced to an external boundary-layer problem over a flat plate or a wedge. Although turbulent flows are, in general, more important, they are usually preceded by laminar flow region. Research for solutions of laminar steady-state forced convection is generally conducted by one of the three privileged directions: direct numerical computation of the boundary-layer equations, differential method based on Blasius analysis, and Pohlhausen's integral method [1-3].

One of the semi-exact methods is the homotopy perturbation method proposed by Ji-Huan He [13]. The applications of this method in different fields of nonlinear equations, integro-differential equations, Laplace transform, and fluid mechanics have been studied by Cai [5], Cveticanin [6], and El-Shahed [7]. The Homotopy perturbation method is a novel and effective method, and has been successfully applied to solve various nonlinear complicated engineering problems that cannot be solved by analytical

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methods [8-11]. Several researches on the behavior of non-Newtonian fluids have been conducted in different geometries by Ghori et al. [8], Ramiar et al. [9], Mahmood et al. [10], and Ariel et al. [11].
The present work deals with the steady laminar forced convection with pressure gradient from a flat plate subjected to a constant temperature with homotopy perturbation method (HPM). The effect of pressure gradient is then studied on velocity and temperature profile and compared with other solutions.

2. Basic idea of homotopy perturbation method

To illustrate the basic ideas of the new method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{1}$$

With boundary conditions

$$B(u, \partial u / \partial n) = 0, \qquad r \in \Gamma \tag{2}$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.
The operator A can, generally speaking, be divided into two parts L and N, where L is linear, while N is nonlinear, Eq. (1), therefore, can be rewritten as follows

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

By the homotopy technique [1, 2], we construct a homotopy $\upsilon(r,P) : \Omega \times [0,1] \rightarrow R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega \tag{4a}$$

Or

$$H(v,p) = L(v) - L(u_0) + pl(u_0) + p[N(v) - f(r)] = 0, \tag{4b}$$

where p € [0,1] is an embedding parameter, u0 is an initial approximation of Eq. (1), which satisfies the boundary conditions. Obviously, from Eq. (4) we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{5}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{6}$$

the changing process of p from zero to unity is just that of v(r, p) from $u_0(r)$ to u(r). In topology, this is called deformation, and $L(\upsilon)$-$L(u_0)$, $A(\upsilon)$-$f(r)$are called homotopic.
In this paper, the present author will first use the imbedding parameter p as a "small parameter", and assume that the solution of Eq. (4) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{7}$$

Setting p=1 results in the approximate solution of Eq. (1):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots. \tag{8}$$

The coupling of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated limitations of the traditional perturbation methods. On the other hand, the proposed technique can take full advantage of the traditional perturbation techniques.

3. Equations

Boundary layer flow over a flat plate is governed by the continuity and the Navier–Stokes equations. Under the boundary layer assumptions and a constant property assumption, the continuity and Navier–Stokes equations become [12]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{9}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dP}{dx}+v\frac{\partial^2 u}{\partial y^2}+gB(T-T_\infty). \tag{10}$$

Under a boundary layer assumption, the energy transport equation is also simplified.

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \tag{11}$$

From Eqs. (10) and (11), the solutions of the energy and momentum equations are coup led. However, the buoyancy force may be neglected if there is a pressure gradient perpendicular to the gravitational force. If the pressure gradient is created by an external force or shape of motion then it isn't neglected. Thus, in the case of the forced convection with a pressure gradient, the solution to the momentum equation is decoupled from the energy solution. However, the solution of the energy equation is still linked to the momentum solution. The following dimensionless variables are introduced in the transformation:

$$\eta=y\sqrt{\frac{U_\infty}{vx}}, \tag{12}$$

$$\theta(\eta)=\frac{T-T_\infty}{T_W-T_\infty}, \tag{13}$$

where θ is nondimensional form of the temperature.

Using Eqs. (9) through (13), the governing equations can be reduced to two equations where f is a function of the similarity variable (η):

$$f'''+\frac{1}{2}ff''=-\lambda,\quad \theta''+\frac{\Pr}{2}f\theta'=0. \tag{14}$$

Where λ and f are related to a pressure gradient and the u velocity by

$$\lambda=-\frac{x}{\rho U_\infty^2}\frac{dP}{dx},\qquad f'=\frac{u}{U_\infty}, \tag{15}$$

The reference velocity is the free stream velocity of forced convection. The boundary conditions are obtained from the similarity variables. For the forced convection case:

$$f(0)=0,\quad f'(0)=0,\quad f'(\infty)=1,\quad \theta(0)=1,\quad \theta(\infty)=0. \tag{16}$$

4. HPM solution for flow over a flat plate

According to Eq.4a and Eq.14:

$$\begin{aligned}&(1-p)(f'''-f_0''')+p(f'''+\frac{1}{2}ff'')=-\lambda,\\&(1-p)(\theta''-\theta_0'')+p(\theta''+\frac{\Pr}{2}f\theta')=0.\end{aligned} \tag{17}$$

We consider f and θ as the following:

$$\begin{aligned}&f=f_0+pf_1+p^2f_2+...\\&\theta=\theta_0+p\theta_1+p^2\theta_2+...\end{aligned} \tag{18}$$

Assuming $f'''=-\lambda$, $\theta''=0$, and substituting f and θ from Eq. (17) into Eq. (18) and some simplification and rearranging based on the powers of p-terms, we have:

$$p^0: \quad f_0'''=-\lambda, \qquad \theta_0''=0, \tag{19}$$

$$f_0(0)=0,\ \ f_0'(0)=0,\ \ f_0'(\infty)=1,\ \ \theta_0(0)=1,\ \ \theta_0(\infty)=0.$$

$$p^1: \quad f_1'''=-\frac{1}{2}f_0f_0'', \qquad \theta_1''=-\frac{pr}{2}f_0\theta_0',$$

$$f_1(0)=0,\ \ f_1'(0)=0,\ \ f_1'(\infty)=0,\ \ \theta_1(0)=0,\ \ \theta_1(\infty)=0. \tag{20}$$

$$p^2: \quad f_2'''=-\frac{1}{2}(f_0f_1''+f_1f_0''), \qquad \theta_2''=-\frac{pr}{2}(f_0\theta_1'+f_1\theta_0') \tag{21}$$

$$f_2(0)=0,\ \ f_2'(0)=0,\ \ f_2'(\infty)=0,\ \ \theta_2(0)=0,\ \ \theta_2(\infty)=0.$$

$$p^3: \quad f_3'''=-\frac{1}{2}(f_0f_2''+f_1f_1''+f_2f_0''),$$
$$\theta_3''=-\frac{pr}{2}(f_0\theta_2'+f_1\theta_1'+f_2\theta_0''), \tag{22}$$

$$f_3(0)=0,\ \ f_3'(0)=0,\ \ f_3'(\infty)=0,\ \ \theta_3(0)=0,\ \ \theta_3(\infty)=0.$$

Solving Eqs. (19)-(25) with boundary conditions (20)–(26), we have:

$$f_0=-\frac{\lambda}{6}\eta^3+b\eta^2 \tag{23}$$

$$f_1=-\frac{\lambda^2}{2520}\eta^7+\frac{b\lambda}{180}\eta^6-\frac{b^2}{60}\eta^5+c\eta^2 \tag{24}$$

$$f_2=-\frac{\lambda^3}{623700}\eta^{11}+\frac{b\lambda^2}{28350}\eta^{10}-\frac{b^2\lambda}{4032}\eta^9+\frac{11b^3}{20160}\eta^8+\frac{c\lambda}{180}\eta^6-\frac{bc}{30}\eta^5+d\eta^2 \tag{25}$$

$$f_3=-\frac{563\lambda^4}{81729648000}\eta^{15}+\frac{563b\lambda^3}{2724321600}\eta^{14}-\frac{7081b^2\lambda^2}{3113510400}\eta^{13}+\frac{577b^3\lambda}{53222400}\eta^{12}-\frac{5b^4}{266112}\eta^{11}+\frac{c\lambda^2}{28350}\eta^{10}-\frac{bc\lambda}{2016}\eta^9+\frac{11b^2c}{6720}\eta^8+\frac{d\lambda}{180}\eta^6-(\frac{bd}{30}+\frac{c^2}{60})\eta^5+e\eta^2 \tag{26}$$

$$\theta_0=g\eta+1, \tag{27}$$

$$\theta_1=pr(\frac{g\lambda}{240}\eta^5-\frac{bg}{24}\eta^4)+m\eta. \tag{28}$$

$$\theta_2=pr((\frac{g\lambda^2}{41472}pr+\frac{g\lambda^2}{362880})\eta^9-(\frac{gb\lambda}{2304}pr+\frac{gb\lambda}{20160})\eta^8+(\frac{gb^2}{504}pr+\frac{gb^2}{5040})\eta^7+\frac{m\lambda}{240}pr\eta^5-(\frac{bm}{24}+\frac{gc}{24}pr)\eta^4)+n \tag{29}$$

$$\theta_3=(\frac{g\lambda^3pr^2}{25159680}+\frac{g\lambda^3pr}{194594400}+\frac{\lambda^3gpr^3}{8626176})\eta^{13}-(\frac{gb\lambda^2pr^2}{967680}+\frac{gb\lambda^2pr}{7484400}+\frac{gb\lambda^2pr^3}{331776})\eta^{12}(\frac{gb^2\lambda pr^2}{2661120}+\frac{gb^2\lambda pr}{887040}+\frac{gb^2\lambda pr^3}{38016})\eta^{11}-(\frac{gb^3pr^2}{43200}+\frac{gb^3pr^3}{12960}+\frac{11gb^3pr}{3628800})\eta^{10}+(\frac{m\lambda^2pr^2}{41472}+\frac{m\lambda^2pr}{362880})\eta^9-(\frac{cg\lambda pr^2}{2304}+\frac{cg\lambda pr}{20160}+\frac{mb\lambda pr}{20160}+\frac{mb\lambda pr^2}{2304})\eta^8+(\frac{cbgpr^2}{252}+\frac{cbgpr}{2520}+\frac{b^2mpr}{5040}+\frac{b^2mpr^2}{504})\eta^7+\frac{n\lambda pr}{240}\eta^5-(\frac{dgpr}{24}+\frac{cmpr}{24}+\frac{nbpr}{24})\eta^4+i\eta \tag{30}$$

Constant coefficient, such as b, c, d, and …, can be calculated via two boundary conditions: $f'(\infty)=1$ *and* $\theta(\infty)=0$

Also, these coefficients are changed via the value of η_∞ which changes with the pressure gradient or variable Pr number. The work of Bird and Cebeci reports values of 5.64 and 8 for η_∞ for a situation when Pr = 1, m = 0 [2, 12]. On the other hand, η_∞ in θ is a function of Pr number as well when there is no pressure gradient. In this condition, we have taken η_∞ to be a function of both m and Pr number in his work.

5. Results and discussion

When there is no pressure gradient, the results are obtained employing HPM as well as the numerical method suggested by Bird for Pr =1 (Table 1) [12]. Also, the θ *(η)* values obtained from the HPM method have very good agreement with the numerical method. Therefore, Nusselt number (*Nu*) can be calculated via this temperature profile.

Fig. 1 shows the variations of f' *(η)* in different pressure gradient (λ) which can be changed between -0.91 to ∞. The minimum value of λ that can be studied by this method is -0.84, because we solve these problems assuming a low pressure gradient.

Table 1. The results of HPM and Numerical methods for *f' (η)* and θ *(η)*

η	**f' (η)**		**θ(η)**	
	HPM	**NM**	**HPM**	**NM**
0	**0**	**0**	**1**	**1**
0.2	0.069907	0.066408	0.930093	0.933592
0.4	0.139764	0.132764	0.860236	0.867236
0.6	0.209441	0.198937	0.790559	0.801063
0.8	0.278723	0.264709	0.721277	0.735291
1	0.347312	0.32978	0.652688	0.67022
1.2	0.414831	0.393776	0.585169	0.606224
1.4	0.480832	0.456262	0.519168	0.543738
1.6	0.544806	0.516757	0.455194	0.483243
1.8	0.606195	0.574758	0.393805	0.425242
2	0.664414	0.629766	0.335586	0.370234
2.2	0.718871	0.68131	0.281129	0.31869
2.4	0.768993	0.728982	0.231007	0.271018
2.6	0.814261	0.772455	0.185739	0.227545
2.8	0.854239	0.81151	0.145761	0.18849
3	0.888611	0.846044	0.111389	0.143955
3.2	0.917222	0.876081	0.082778	0.123918
3.4	0.940107	0.901761	0.059893	0.088239
3.6	0.957524	0.92333	0.042476	0.06667
3.8	0.969974	0.941118	0.030026	0.058882
4	0.978212	0.955518	0.021788	0.031482
4.2	0.983235	0.966957	0.016765	0.033043
4.4	0.986244	0.975871	0.013756	0.024129
4.6	0.988579	0.982684	0.011421	0.017317
4.8	0.991602	0.98779	0.008398	0.012211
5	0.996533	0.991542	0.003467	0.008458

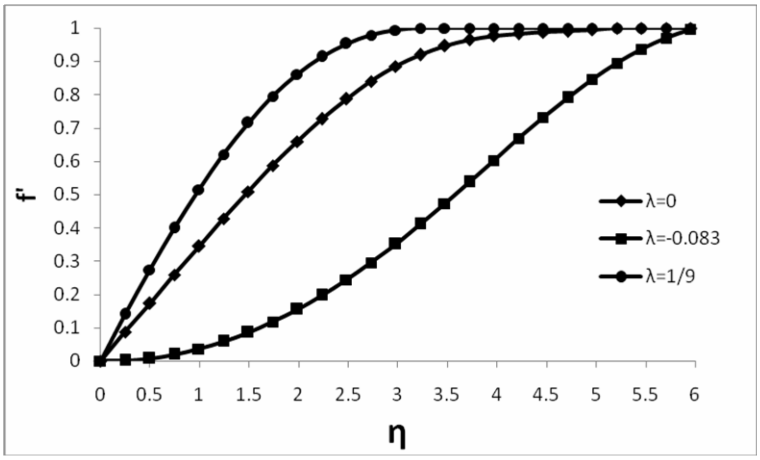

Figure 1. The results of HPM methods for f' (η) for different pressure gradient

Fig. 2 suggests that for a fixed value of η the ratio of hydrodynamic boundary layer thickness to thermal boundary layer thickness increases by increasing Pr number. This effect is more pronounced for higher pressure gradients (higher values of λ).

The effect of pressure gradient variation on the thermal boundary layer thickness is higher than the Pr number, and the thermal boundary layer thickness is decreased via increasing pressure gradient. In other words, it increases the tendency of the fluid temperature to reach the surface temperature.

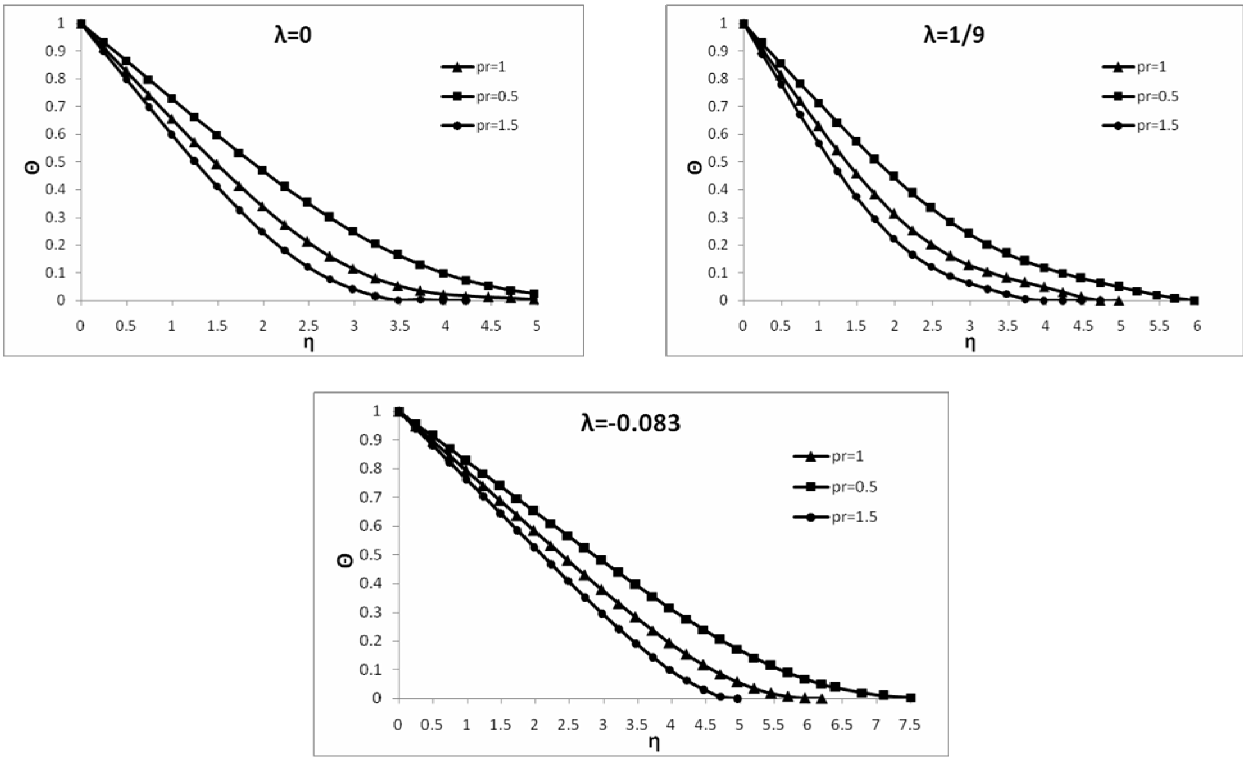

Figure 2. The results of HPM and NM methods for θ (η) and HPM method for pressure gradient

6. Conclusions

In this paper the heat transfer problem, the equations of momentum and energy, were solved with the homotopy perturbation method over a flat plate. Results are in good agreement with those obtained numerically. Also, the velocity and temperature profiles were obtained as a function of η, pr number, and pressure gradient λ. Using HPM method, the range of permissible pressure gradient (λ) was obtained as -0.083 to 0.111. Finally, an attempt has been made to show the effect of pressure gradient and *Pr* number in hydrodynamic and heat transfer boundary layer.

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