Iranian Journal of Chemical Engineering
Vol. 9, No. 4 (Autumn), 2012, IAChE

Instability Analysis of Miscible Displacements in Homogeneous Porous Media

M. R. Shahnazari[1], F. Eslami[2], Sh. Rezazadeh[3]*
1- Assistant Professor of Mechanical Faculty, K.N.Toosi University of Technology
2- Department of Mech. Eng , K.N.Toosi University of Technology
3- Planning Engineer & Project Manager, Aquarion Water Company

Abstract
Stability analysis of miscible displacement has several applications in industries such as oil recovering and ground water tables. In this article an analytical solution is presented based on Tan and Homsy's results for stability analysis in t = 0. Moreover, a novel semi analytical solution is used, based on weighted residual method, to solve the Fourier space equations. The results are shown as σ (disturbance growth rate) – k (wave number); profiles for different values of mobility ratios and times. A comparison with the results of the other researches is also presented.

Keywords: *Stability Analysis, Displacement, Miscible Fluid, Porous Medium*

1. Introduction

Fluid flow instability is one of the important and classic problems of Fluid Mechanics. The transition from laminar flow to turbulent flow and instability of the fluid interface has become one of the most common problems of fluid flow instability in recent years.

Investigations of the instability of miscible or immiscible displacement has several applications in the industries with regards to the fluid flow in porous media. Therefore, this subject has been taken seriously by the researchers since the 1950s. This phenomenon was modeled for the first time by Hill, in 1952 [1]. When a fluid with lower viscosity replaces a fluid with higher viscosity, the interface of the two fluids would commonly get perturbed. The development of the perturbation will cause a phenomenon which is called Viscous Fingering [2].

The most important application of this phenomenon is in the efficient oil recovery. In many applications, the viscous fingering instabilities are undesirable as they result in early breakthrough. Also, any design aimed towards eliminating the fingering instabilities or controlling the growth of the viscous fingers is of technological importance. Hill and Suffman-Taylor in the 1960s and

∗ Corresponding author: shahnazari@kntu.ac.ir

Homsy, in recent years, have studied the effect of the different parameters on the viscous fingerings. However, because of the numerical solution complexities, most of these investigations were based on the experimental data [3]. Displacement processes in porous media are of special consideration in efficient oil recovery, packed columns, fixed bed regeneration, etc.

Generally, the lower viscosity of the displacing fluid towards the displaced fluid would lead to the hydrodynamic instability which results in a channel formation of the displacing fluid in the displaced fluid region [4].

The first mathematical analysis of linear instability for the displacement of two immiscible fluids was done by Chouke et al. in 1959 [5]. Considering the surface tension at the interface, they figured there is a dangerous cut-off wave length for instability. Development of their theory for the miscible fluids in the absence of surface tension and diffusion explains the net flux of molecules from a region of higher concentration to one of lower concentration, resulting in the borderless increase of growth rate with the wave number. This phenomenon is physically unreal and there are physical mechanisms which would dampen the short wavelengths.

Perrine clarified that the diffusion effect should be considered for the miscible fluids, but his solution for the linear stability equations was not accurate. Inserting the diffusion effect leads to the time dependence of the base state solution and results in errors and complexity [6].

Heller also investigated the miscible displacements including diffusion. Nevertheless, he considered some uncertain assumptions about the type of perturbations which lead to a second order, non-homogeneous Eigenvalue equation [7]. With regards to a system described by linear stability theory, there should not be any indication of
the non-homogeneity, therefore, Heller’s method is wrong.

Schowalter [8] studied the fingering phenomenon due to both the density and viscosity. He used an interpenetrated region with constant thickness for diffusion computation. He assumed a constant mass flux to achieve a base state. This assumption reduces the problem to some particular concentration profile which is not real because of the time dependency of the base state solution.

Wooding was one of the few researchers who studied the stability of the time reliance based state [9]. He considered the problem as an initial value problem and expanded the perturbation solution by Hermit polynomials. His main focus was on gravitational instability, therefore he used Boussinesq approximation. He found that even though the small perturbations should tend to zero at $t \to \infty$, they can grow in a wavelength range, as there would be the possibility of offending from the linear confinement theory.

Tan and Homsy [3] considered a system with exponential viscosity profile and isotropic dispersion ($D_{\parallel} = D_{\perp}$). Dispersion is the process whereby solutes are mechanically mixed during advective transport caused by the velocity variations at the microscopic level. The results showed that at times that are long enough, dispersion makes a shift in the bigger wavelengths and leads to stability of the flow. Zimmerman and Homsy [10] performed several experiments to study spreading, tip splitting, shielding and pairing phenomena. According to their results, pairing occurs in the adjacent fingers. Also, growth of a finger may stop the other fingers from growing, which is called Shielding. The results show the sensitivity and complexity of two dimensional fingering. The finite difference method (FDM) is a standard method for numerical simulation of the miscible displacement in the porous medium, regardless of the dispersion effects. Rogerson and Meiberg [11] investigated the similarities of vertical and horizontal displacements in viscous fingering. Manickam and Homsy [12] discovered that the non-uniformity of the viscosity-concentration profile is effective in the non-linear growth of the viscous fingers. Riaz and Meiberg [13] performed a numerical-experimental analysis of viscous fingering in a radial flow Hele-Shaw cell. Brailovsky et al. [14] presented an analytical solution introducing a curve linear coordinate. This was the first time that a curve linear system was used to analyze the fingering phenomenon, nevertheless, the last assumptions in other researches were confirmed. Dewit [15] studied the viscous fingering regarding a surface reaction at the interface of the two fluids. In this study the reaction was assumed to be of third order. The base state solution was achieved for high Damkohler numbers (Da). In another investigation, Dewit used the Adams-Bashforth method to analyze the governing equation which includes the non-linear terms. The Fourier space equation has been studied linearly. In this article the linear stability of miscible displacements has been investigated. After the problem simulation, it would be analytically solved for t=0. In addition, for $t \neq 0$, a new semi-analytical solution has been used and the results have been presented.

2. Theory

The considered system can be seen in Fig. (1). With the velocity of U and in the x direction the flow is assumed to be uniform and incompressible. The porous medium is homogeneous.

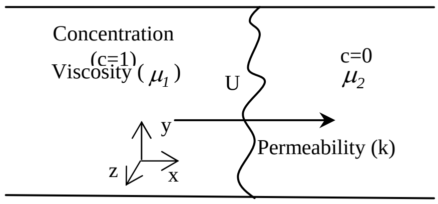

Figure 1. Schematic of the miscible displacement in a porous medium

The porosity and dispersion are constant and isotropic.
The governing equations can be written as:

$$\nabla . u = 0 \tag{1}$$

$$\nabla p = -\mu u \tag{2}$$

$$\frac{\partial c}{\partial t} + u.\nabla c = D\nabla^2 c \tag{3}$$

Note: u is the volume averaged velocity, p is the pressure, μ is the ratio of the viscosity to the permeability of the medium referred to simply as the "viscosity", c is the concentration of the displacing fluid and D is the isotropic dispersion coefficient taken throughout to be constant. Equations (1-3) are continuity, Darcy and concentration equations respectively. The viscosity is assumed to vary exponentially with the concentration:

$$\frac{1}{\mu}\frac{d\mu}{dc} = R \tag{4}$$

$$\alpha = \exp(R) \tag{5}$$

Where R is a parameter determined by mobility ratio and $\alpha = \frac{\mu_2}{\mu_1}$.

Regarding the constant velocity of the fluid, we consider a moving reference $x = x_1 - Ut$ to assume the interface motionless (x_1 is the flow direction in fixed coordinate and x is the flow direction in moving coordinate).

$$u_x + v_y + w_z = 0 \tag{6}$$

$$P_x = -\mu u - \mu U \tag{7}$$

$$P_y = -\mu v \tag{8}$$

$$P_z = -\mu w \tag{9}$$

We transform the system of equations to Moving Coordinate System with the velocity U and the dimensionless equations with $U, D/U, D/U^2$, μ_1 as velocity, length, time and viscosity scales relatively. So we have:

$$P_x = -\mu u - \mu \tag{10}$$

$$c_t + uc_x + vc_y + wc_z = \nabla^2 c \tag{11}$$

The time dependent base state solution to the dimensionless equations in the Moving Coordinate System is:

$$u = v = w = 0 \tag{12}$$

$$\mu_0 = \mu_0(c_0) = \mu_0(x,t) \tag{13}$$

$$c_0(x,t) = \frac{1}{2}[1 + erf\left(\frac{x}{2\sqrt{t}}\right)] \tag{14}$$

$$P_0(x,t) = -\int^{x} \mu_0(s,t)ds \tag{15}$$

3. Stability analysis

Linear stability analysis of the above problem could be performed by transforming the equation into Fourier space. The perturbation parameters would be introduced as [3]:

$$(u',c') = (\delta_u, \delta_c) exp(\sigma t) exp[i(k_y y + k_z z)] \tag{16}$$

Using quasi-steady state approximation (QSSA), which is the higher variations of perturbation rate in comparison with the variation rate of base state [5], the Fourier space equations can be written as:

$$(\frac{d^2}{dx^2} + \frac{1}{\mu_0}\frac{d\mu_0}{dx}(x,t_0)\frac{d}{dx} - k^2)\delta_u = (\frac{k^2}{\mu_0(x,t_0)})(\frac{d\mu}{dc})\delta_c \tag{17}$$

$$(\frac{d^2}{dx^2} - \sigma - k^2)\delta_c = -\frac{dc_0}{dx}(x,t_0)\delta_u \tag{18}$$

Noting equation (4) and incorporation of equations (17) and (18) leads to Fourier space equation according to velocity perturbation parameter [3]:

$$(\frac{d^2}{dx^2}-\sigma(t_0)-k^2)(\frac{d^2}{dx^2}+R\frac{dc_0}{dx}(x,t_0)\frac{d}{dx}-k^2)\delta_u$$
$$=Rk^2\frac{dc_0}{dx}(x,t_0)\delta_u \quad (19)$$

Equation (19) is the Eigenvalue problem. The perturbations become zero at $x \to \mp\infty$. At $t=0$, c_0 can be expressed as a step function so $\frac{dc_0}{dx}(x,t_0)=\delta(x)$.

3-1. Stability analysis at t=0

Stability analysis for a step function concentration profile was first studied by Chouke [5]. In this paper, the analysis method at $t=0$ is similar to the Tan and Homsy's [9]. The equation for step function concentration profile (19) is simplified as:

$$(\frac{d^2}{dx^2}-\sigma-k^2)(\frac{d^2}{dx^2}-k^2)\delta_u=0 \quad (20)$$

According to the boundary condition $\delta_u=0$ at $x \to \mp\infty$, the answer can be written as:

$$\delta_u=A_1 exp(lx)+B_1 exp(kx), \quad x<0 \quad (21\text{-a})$$

$$\delta_u=A_2 exp(-lx)+B_2 exp(-kx), \quad x>0 \quad (21\text{-b})$$

where $l^2=\sigma+k^2$. As for the Delta function property, the comparative conditions would be:

$$\delta_u(0^+)=\delta_u(0^-) \quad (22\text{-a})$$

$$\alpha\frac{d\delta_u}{dx}(0^+)=\frac{d\delta_u}{dx}(0^-) \quad (22\text{-b})$$

$$\delta_c(0^+)=\delta_c(0^-) \quad (22\text{-c})$$

Integration of equation (19) from 0^- to 0^+ leads to:

$$\int_{0^-}^{0^+}(\frac{d^2}{dx^2}-\sigma-k^2)(\frac{d^2}{dx^2}+k\delta(x)\frac{d}{dx}-k^2)\delta_u dx$$
$$=Rk^2\int_{0^-}^{0^+}\delta(x)\delta_u dx \quad (23)$$

Equations (22-a) to (22-c) and (23) are necessary conditions for continuity, velocity disturbance term, pressure and concentration, respectively. Equations (22) and (23) can be shown in the matrix form:

$$AX=0 \quad (24)$$

Where $X=\begin{bmatrix}A_1\\B_1\\A_2\\B_2\end{bmatrix}$ represents the unknown vector of perturbation velocity profile (eq.21) for $x>0$ and $x<0$. According to the equations (22) and (23), the coefficient matrix A is:

$$A=\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & 0 & -1 & 0\\ l & k & \alpha l & \alpha k\\ F_1 & F_2 & F_3 & F_4\end{bmatrix} \quad (25)$$

Where $F_1=k^2l-R(k^2+l^3)$, $F_2=kl^2-Rk^2-Rkl^3$, $F_3=k^2l$ and $F_4=kl^2$ are the coefficients resulted from the integration of equation (23).

By equating the determinant of matrix A to zero, the characteristic equation is:

$$2lk+2k^2-Rk=0 \quad (26)$$

or $l = \frac{-k \mp \sqrt{k^2 + 2Rk}}{2}$ and thereupon:

$$\delta_u = \frac{1}{2}[(Rk - k^2) - k\sqrt{k^2 + 2Rk} \quad (27)$$

Therefore, the most unstable case would be:

$$\sigma_m = [\frac{(-2+\sqrt{5})(7-3\sqrt{5})}{3-\sqrt{5}}]\frac{R^2}{4} @ k_m$$

$$= \frac{(2\sqrt{5}-4)R}{4} = 0.118R \quad (28)$$

3-2. QSSA at t>0

For $t > 0$ the concentration profile would be a function of location at a constant time. Regarding the base state solution as $c_0 = \frac{1}{2}[1 + erf\left(\frac{x}{2\sqrt{t}}\right)]$ and equation (14), a simple analytical solution could not be found for equation (19) [14]. In this research, the weighted residual method (WRM) has been used, noting the analytical solution of equation (19) at t=0, the answer is assumed to be:

$$\delta_u = \sum a_j^+ exp(+jlx) + \sum b_j^+ exp(+jkx), \quad x < 0$$

$$\delta_u = \sum a_j^- exp(-jlx) + \sum b_j^- exp(-jkx), \quad x > 0 \quad (29)$$

Where the boundary condition $\delta_u = 0$ at $x \to \mp\infty$ is satisfied. In order to determine the a_j and b_j, the profile should satisfy the equation (19). Moreover, for $x \in]-\infty, 0]$ and $x \in [0, \infty[$, matching conditions have been considered as supplementary equations in order to preserve the continuity of velocity perturbation, pressure and concentration. Also, the final equation has been achieved by integration of the equation (19) from 0^- to 0^+. Thus, in order to determine $2N+4$ unknown coefficients ($j = 0 \quad to \quad N+1$), 4 matching equations were used in addition to the $2N$ chosen nodes.

4. Results and discussion

The disturbance growth rate is illustrated in Fig. (2), showing the relationship between σ and k. The values are presented for $t = 0$, $R = 3$ and $\alpha = 20.09$. It can be determined that for wave numbers of greater than 0.75, the stability conditions would be confirmed. Also, the most critical situation occurs at $k = 0.345$ where the growth rate is equal to 0.202. The disturbance growth rate for $t \geq 0$ and $R = 3$ is shown in Fig. (3).

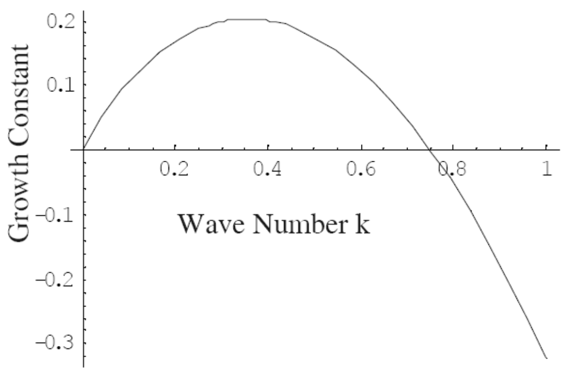

Figure 2. The growth constant versus wave number for t=0, R = 3 and α=20.09

When time advances, the amount of critical σ decreases and the range of k becomes slender where the instability occurs. Figs. (4) and (5) show the relationship between the characteristic growth rate and the wave number for different times and the two values of R=2 and R=5.

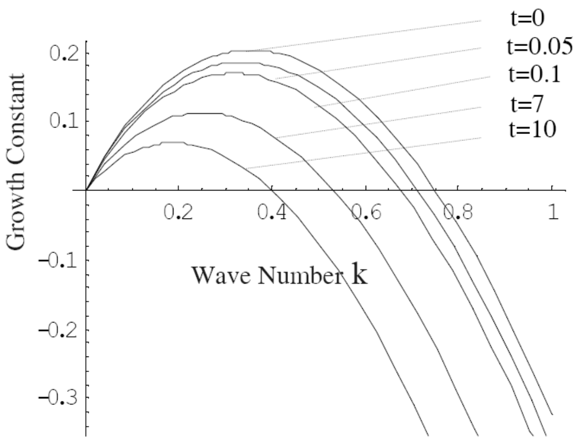

Figure 3. The growth constant versus wave number for t=0 and R = 3

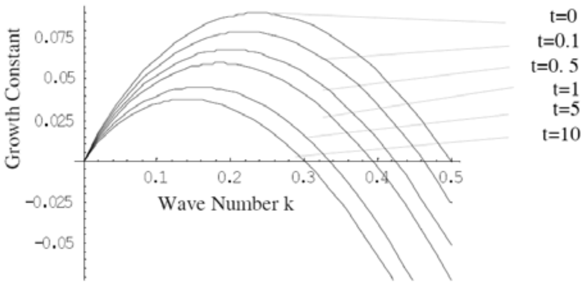

Figure 4. Relation between characteristic growth rate and wave number for different times and R = 2 and R =5

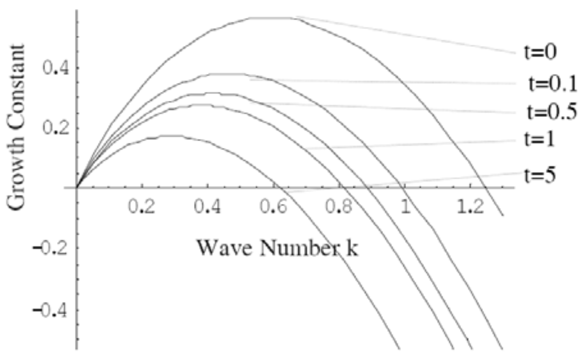

Figure 5. Relation between characteristic growth rate and wave number for different times and R =5

The dependence of σ_m and k_m on the mobility ratio are approximately proportional to R^2 and R respectively, which can be recalled from Analytical Solution.

Fig. (6) shows the most critical value of the growth rate according to time and for different R values. The growth constant decays with time. For R<0.2 and time >2, the decay rate of growth constant is negligible.

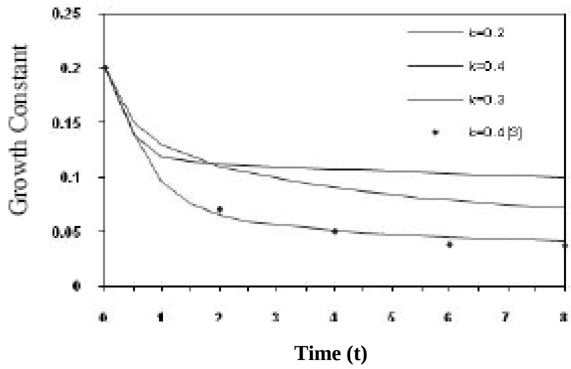

Figure 6. The most critical value of growth rate according to time and for different *R* values

5. Conclusions

Above, miscible displacement stability in a porous medium was studied. Presenting a semi stable model, the analytical solution for $t = 0$ was expounded while a semi analytical method was used in accordance to WRM. The investigations lead to prediction of a range for wave number where the instability conditions arise. For each mobility ratio and by the advancement of time, this range becomes narrower while the value of the most critical σ decreases. Also, the wave number range for instability and the most critical σ increase proportionally to R by the increase in mobility ration every time.

Comparing these results with the Tan and Homsy's shows good agreement which proves the efficiency of the introduced method.

References

[1] Hill, S., "Channeling in packed columns", Chem. Eng. Sci., 1, 247–253 (1952).

[2] Sahimi, M., Flow and transport in porous media and fractured rock, VCH, Weinheim, Germany (1995).

[3] Tan, C. T. And Homsy, G. M., "Stability of miscible displacements in porous media: Rectilinear flow", Phys. Fluids, 29, 3549–3556 (1986).

[4] Jerauld, G. R., Davis, H. J. And Scriven, L. E., "Stability of fronts of permanent form in immiscible displacement", SPE 13164, Soc. Petr. Eng., Dallas (1984).

[5] Chouke, R. L., Van Der Meurs, P. and Van Der Poel, C., "The stability of a slow, immiscible, viscous liquid-liquid displacement in a permeable media", Trans AIME, 216, 188–194 (1959).

[6] Perrine, R. L., "A unified theory for stable and unstable miscible displacement", SPE 543 (1963).

[7] Heller, J. P., "Onset of instability patterns between miscible fluids in porous media", J. Appl. Phys., 37, 1566–1579 (1965).

[8] Schowalter, W.R., "Stability criteria for miscible misplacement of fluid from a porous medium.", AICHE J., 11, 99 (1965).

[9] Wooding, R. A., "Free convection of fluid in a vertical tube filled with porous material", Fluids Mech., 13, 129–144 (1962).

[10] Zimmerman, W. B. And Homsy, G. M., "Nonlinear viscous fingering in miscible displacement with anisotropic dispersion", Phys. Fluids a Fluid, 3, 1859–1872 (1991).

[11] Rogerson, A. and Meiburg, E., "Shear stabilization of miscible displacement processes in porous media", Phys. Fluids a Fluid, 5, 1344–1355 (1993).

[12] Manickam, O. and Homsy, G.M., "Simulation of viscous fingering in miscible displacements with non-monotonic viscosity profiles." Phys. Fluids, 6, 95–107 (1994).

[13] Riaz, A., Pankiewitz, E. and Meiburg, E., "Linear stability of radial displacements in porous media: Influence of velocity-induced dispersion and concentration-dependent diffusion", Phys. Fluids, 16(10), 3592–3598 (2004).

[14] Brailovsky, I., Babchin, A., Frankel, M. and Sivashinsky, G., "Fingering instability in water-oil displacement", Transport in Porous Media, 63, 363–380 (2006).

[15] De Wit, A., "Fingering of chemical fronts in porous media.", Phys. Rev. Lett, 87, 054502/1–4 (2001).

[16] De Wit, A., De Kepper, P., Benyaich, K., Dewel, G. And P. Borckmans, "Hydrodynamic instability of spatially extended bistable chemical systems", Chem. Eng. Science, 58, P.4823-4831 (2003).