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A New Model for Prediction of Heat Eddy Diffusivity in Pipe Expansion Turbulent Flows

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Abstract
A new model to calculate heat eddy diffusivity in separating and reattaching flows based on modification of constant Pr_t is proposed. This modification is made using an empirical correlation between maximum Nusselt number and entrance Reynolds number. The model includes both the simplicity of Pr_t=0.9 assumption and the accuracy of two-equation heat-transfer models. Furthermore, an appropriate low Reynolds number $k-\varepsilon$ model is adopted for calculation of eddy viscosity. The model is used for prediction of Nusselt number distribution at various ranges of Reynolds number and expansion ratio. The numerical results are compared with available experimental data in the literature and have shown good agreement. The CPU time for the present model is about 33% less than that of two-equation heat-transfer model.

Keywords: *Expansion Flow, Heat Eddy Diffusivity, Low Reynolds Number $k-\varepsilon$ Model, Nusselt Number, Turbulent Prandtl Number*

Introduction

An accurate prediction of heat transfer downstream of backward-facing step is important in many engineering applications such as combustors, nuclear reactors and heat exchangers. Separation and reattachment in sudden expansion flows not only affect the flow field structure but also influence the mechanism of heat transfer. In particular, around the reattachment point, heat transfer increases several times with respect to the fully-developed region.

In most of the previous studies, the $k-\varepsilon$ model has been used to predict separating and reattaching turbulent flows. Although the standard $k-\varepsilon$ model is quite useful for many flows, some major problems still remain for other flows. For example:

1. The standard $k-\varepsilon$ model usually gives 15-20% under-prediction of the flow reattachment length downstream of a backward-facing step, which is the most fundamental quantity to be predicted in separating and reattaching flows [1].
2. The Nusselt number predicted by the standard $k-\varepsilon$ model is not usually in good agreement with the experimental data [2].

In the standard $k-\varepsilon$ model, wall functions are employed as the boundary conditions on solid walls. However, the application of wall functions for the recirculation regions is open to question.

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Low-Reynolds-number $k-\varepsilon$ models, in contrast to $k-\varepsilon$ model, are applicable in low Reynolds number regions. They do not need wall function implementation. In recent years, by using direct analytical solution and invoking experimental measurement, significant advances have been achieved in these models. Hsieh, *et al*. [3] examined nine low-Reynolds-number $k-\varepsilon$ models in pipe flow with sudden expansion. They found that except for models of Abe, *et al*. (AKN[1] model, [4]) and Chang, *et al*. (CHC[2] model, [5]), no model is able to correctly predict the wall heat transfer. This is due to the lack of near-wall limiting behavior or existence of singularity at reattachment points in those models.

Although AKN and CHC models are quite useful for the prediction of velocity field, they over-predict the Stanton number as high as 30%. This is due to the Pr_t= 0.9 assumption [6]. Turbulent Prandtl number in boundary layer has a value around 0.9 but due to the dissimilarity between velocity and temperature fields in rotational and reattachment regions, this assumption is not valid there.

The accurate prediction of heat transfer in a complex flow needs an accurate prediction of Pr_t. Recently, research works are conducted to calculate turbulent heat transmission using two-equation models [6-9]. The predictions of Stanton number behind a step with the two-equation model by Abe, *et al*. [6] are in good agreement with experimental data. The major difficulties in these models are related to their computational complexity and CPU time.

In this study, a new model to calculate heat eddy diffusivity in separating and reattaching flows based on modification of constant Pr_t is proposed. This modification is made using an empirical correlation between maximum Nusselt number and entrance Reynolds number. The results show that the present model is capable of predicting the heat transfer in pipe expansion turbulent flows quite successfully. The Nusselt number distribution is almost in perfect agreement with experiments at various ranges of Reynolds number and expansion ratio.

1. Abe, Kodoh and Nagano
2. Chang, Hsieh and Chen

Governing Equations

In order to simplify the problem and thus facilitating the numerical simulation, the following assumptions were made on Favre-averaged Navier-Stokes equations.

1. The flow is steady, incompressible, and axisymmetric.
2. Turbulent flux terms are modeled based on the Bousinesque approximation.
3. Radiative heat transport is neglected.

Under the above assumptions, the governing equations can be written as:

$$\frac{1}{r}\left[\frac{\partial}{\partial x}(r\rho U\phi)+\frac{\partial}{\partial r}(r\rho V\phi)\right.$$
$$\left.-\frac{\partial}{\partial x}\left(r\Gamma_\phi\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial r}\left(r\Gamma_\phi\frac{\partial\phi}{\partial r}\right)\right]=S_\phi \tag{1}$$

where Φ stands for the dependent variables (1, U, V, k, ε, h_t, …) being considered, Γ_Φ is the effective diffusion coefficient, and S_Φ is the source term (Table 1). The total enthalpy h_t is defined as:

$$h_t=\int_{T_0}^{T}c_P dT+\frac{U^2+V^2}{2}+k \tag{2}$$

The equation of state is written as:

$$P=\rho T\frac{R_0}{M} \tag{3}$$

The eddy viscosity ν_t is related to k and ε through the Kolmogorov-Prandtl relation as:

$$\nu_t=C_\mu f_\mu\frac{k^2}{\varepsilon} \tag{4}$$

Experimental constants and model functions in turbulent models are selected such as to modify asymptotic behavior of variables in the vicinity of solid wall and also compensate the weakness of the standard $k-\varepsilon$ models for low Reynolds numbers. In the present work, the CHC turbulent model is employed for the flow field [5]. In Table 2 constants of CHC model are presented. The model functions used in CHC model are as follows:

$$f_1 = 1.0 \tag{5}$$

$$f_2 = [1-exp(-0.063y^*)][1-0.01exp(-R_t^2)]$$

$$f_\mu = \left[1-exp(-0.0215y^*)\right]^2 (1+31.66R_t^{-5/4})$$

In these functions, y* and R_t are dimensionless distance from the wall and local turbulent Reynolds number, respectively, and are defined as follows:

$$y^* = \frac{u_k y}{\nu} \tag{6}$$

$$R_t = \frac{k^2}{\nu\varepsilon} \tag{7}$$

where u_k is a velocity scale for turbulent flow and is defined as follows:

$$u_k = \sqrt{k} \tag{8}$$

α_t, the heat eddy diffusivity, is usually modeled using the following relation:

$$\alpha_t = \frac{\nu_t}{Pr_t} \tag{9}$$

To calculate turbulent Prandtl number (Pr_t), turbulent shear stress ($\overline{uv}$), turbulent heat flux ($\overline{tv}$), velocity gradient, and temperature gradient should be measured. Due to the difficulties in measuring these quantities, there are scatterings in the experimental data [10]. In the two-equations heat model, turbulent diffusion coefficient is calculated using the following equation [6]:

$$\alpha_t = \left[\frac{k^2}{\varepsilon}\left(\frac{0.2R}{0.5+R}\right)+0.3k^{1/2}\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}\frac{(2R)^{1/2}}{Pr}exp\left(-\frac{R_t^2}{200^2}\right)\right]$$
$$\left[1-exp\left(-\frac{y^{**}}{14}\right)\right]\left[1-exp\left(-\frac{Pr^{1/2}y^{**}}{14}\right)\right] \tag{10}$$

In this equation, y^{**} is the dimensionless distance from wall and is defined as:

$$y^{**} = \frac{u_\varepsilon y}{\nu} \tag{11}$$

where u_ε is Kolmogorov velocity scale and is defined as:

$$u_\varepsilon = (\nu\varepsilon)^{1/4} \tag{12}$$

Equations (9) and (10) have the same basis except that the effects of wall are accounted for in equation (10) by introducing the parameters y** and R_t. Furthermore, instead of Pr_t an expression in terms of R is used ($R \propto 1/Pr_t$). The assumption of constant R in equation (10) is equivalent to the assumption of constant Pr_t. R is the ratio of time scale in temperature domain to that of velocity domain and is defined as:

$$R = \frac{k_t/2\varepsilon_t}{k/\varepsilon} \tag{13}$$

where k_t is the temperature fluctuation variance, and ε_t is the dissipation rate of k_t. Therefore, to calculate R, two transport equations $k_t-\varepsilon_t$ should be added to $k-\varepsilon$ equations [6]. The two-equation heat transfer model $k_t-\varepsilon_t$ can predict heat transfer in separating and reattaching flow regions downstream of a backward-facing step. However, this model can not be applied universally. Furthermore, the use of $k-\varepsilon$ and $k_t-\varepsilon_t$ greatly increases the computational complexity and time.

Figure 1 shows that the assumption of Pr_t=0.9 over-predicts Stanton number (St=Nu/Re Pr). Although the results based on Pr_t=0.9 do not coincide with the experimental data, but it follows almost the same trend. This means that the assumption of constant Pr_t in the whole flow domain is acceptable. Furthermore, a proper estimate of Pr_t or R in the flow behind a step decreases the discrepancy between numerical and experimental Nusselt number distributions.

The experimental measurements behind a step for various Reynolds numbers and expansion ratios [11-13] show that the maximum Nusselt number behind a step is only a function of inlet Reynolds number Re_d. The best curve fit to experimental data is expressed as [11, 13]:

$$Nu_{max} = 0.2\,Re_d^{2/3} \tag{14}$$

Figure 1 shows that the variation of R changes the maximum Stanton number ($Nu_{max} = f(R)$). Therefore, it seems that there is a relation between R and Re_d ($R = f(Re_d)$). One may suggest a relation similar to equation (14) but with two unknown constants:

$$R = m\,Re_d^n \tag{15}$$

Determination of m and n requires known values of R for two different values of Re_d.

Numerical Method

The numerical method used in the present study is based on the SIMPLE algorithm [14]. The conservation equations are discretized using the finite volume approach based on collocated grids and power-law scheme. The discretized equations are solved iteratively using a line by line solution method in conjunction with a tridiagonal matrix algorithm [14]. A non-uniform grid arrangement is used in the present computations. The grid is systematically refined to find grid-independent results. The results appear to be grid-independent with a 151×101 mesh cells in the axial and radial directions, respectively. Due to the presence of a large velocity gradient and turbulent kinetic energy in the vicinity of the wall, the grid should be sufficiently fine to let the first node to be within the viscous sub layer [3]. The criterion for convergence is

$$max\left|\frac{\phi^i-\phi^{i-1}}{\phi_{max}^{i-1}}\right| < 10^{-4} \tag{16}$$

where the superscript i denotes the number of iterations and the subscript “max” refers to a maximum value over the entire field of iterations.

Results and Discussions

To study the accuracy of the developed model, experimental measurements of Baughn, *et al*. [12, 13] are used. In the work of Baughn, *et al*. no data were given for the entrance. However, they reported that a fully-developed profile of mean velocity and a low level turbulence intensity were observed in the inlet region because the upstream tube was sufficiently long. Therefore, the mean axial velocity profile was assumed using the one-seventh law and the mean radial velocity was assumed to be zero. The inlet profiles for k and ε were given by the following empirical equations [3]:

$$k_{in} = 0.003\,U_{in}^2 \tag{17}$$

$$\varepsilon_{in} = 2C_\mu\, k_{in}^{1.5}/0.03d \tag{18}$$

where d and U_{in} are diameter and mean axial velocity at the inlet, respectively. Inlet air temperature was kept constant at 300 K.

At the exit of the flow field, there are no conclusive boundary conditions. The only remedy is to assume zero gradients in x-directions. In order to minimize the effects of this assumption on the results, the computational domain downstream of the step should be considered long enough (almost 20 times the step height).

Due to no-slip and no-permeability boundary conditions on the wall, $U=0$, $V=0$, $k=0$, and $\varepsilon = \nu \partial^2 k / \partial r^2 \big|_W$ [4]. Wall temperature downstream of the step, in the experimental setup, is kept at 310 K. At the symmetry axis, the condition of zero gradient in r-direction is met except for the radial velocity component, which is naturally zero.

The first condition to fix m and n in equation (15) is to assume Re_d = 17310 with d/D= 0.4. In turbulent boundary layer, where there is a similarity between velocity and temperature fields, R will be held equal to 0.5 [6]. For the first try, we assume R = 0.5. The variation of Nusselt number is shown in Figure 2, the curve of which is above the experimental data. The reason for this behavior is that the mean velocity gradient, $\partial U / \partial y$, almost vanishes at reattachment point. On the other hand, the mean temperature gradient, $\partial T / \partial y$, is similar to that of a flat plate. As a consequence, no similarity exists between the velocity and the temperature fields in backward-facing step flow.

Near the wall, R can be expressed as follows [6, 10]:

$$R \propto \frac{\overline{tv}\,\dfrac{\partial U}{\partial y}}{\overline{uv}\,\dfrac{\partial T}{\partial y}} \tag{19}$$

Therefore, we expect a value smaller than 0.5 for R in backward-facing step flow. By decreasing R to 0.3 and repeating the computations, the topology of the curve, as shown in Figure 2, remains almost the same but it moves down slightly. After some trial and error, the results with R = 0.17 agree with the experimental data. The fully-developed Nusselt number, Nu_{fd}, inside the pipe, is computed the using Dittus-Boelter experimental relation:

$$Nu_{fd} = 0.023\, Re_D^{0.8}\, Pr^{0.4} \tag{20}$$

To investigate the accuracy of results, the velocity and temperature profiles are compared with the experimental data in Figures 3 and 4, respectively. The results are sufficiently accurate. As addressed in reference [5], the difference within the recirculation region is due to measurement errors. After checking the accuracy, computations are repeated for Re_d = 44540 and the expansion ratio equal to 0.4. For this case, the best Nusselt number distribution is obtained with R= 0.07 (Figure 5). Summing up the two sets of calculations, the values of m and n in equation (15) are obtained to be 3877 and -0.94, respectively.

Using equation (15) to estimate R, there is no need to solve k_t and ε_t equations any more. Hence, the computational time reduces by up to 33 percent. To investigate the accuracy and the validity of the proposed model, several computations at different Reynolds numbers and expansion ratios are performed. As shown in Figure 6, the present model is sufficiently accurate at various expansion ratios and Reynolds numbers.

Conclusion

A new model to calculate heat eddy diffusivity in separating and reattaching flows based on modification of constant Pr_t has been proposed. This modification is made using an empirical correlation between maximum Nusselt number and entrance Reynolds number. The model includes both the simplicity of Pr_t = 0.9 assumption and the accuracy of two-equation heat-transfer models. Furthermore, an appropriate low-Re $k-\varepsilon$ model has been adopted for the calculation of the eddy viscosity. This model was calibrated using the experimental data of Baughn, *et al*. The performance of this model was tested at various ranges of flow Reynolds number and expansion ratio in a pipe. The numerical results have also been compared with the available experimental data in the literature and a good agreement is observed. The complexity of the present model is much less than the two-equation heat transfer model. The corresponding CPU computation time for the present model is about 33 percent less than the two-equation heat-transfer model.

Table 1. Transfer coefficients and source terms in governing equation

ϕ	Γ_ϕ	S_ϕ
1	0	0
U	μ_e	S_U
V	μ_e	$S_V - 2\mu_e V/r^2$
k	$\mu + \mu_t/\sigma_k$	$G - \rho\varepsilon$
ε	$\mu + \mu_t/\sigma_\varepsilon$	$\left(C_{\varepsilon 1} f_1 \varepsilon G - C_{\varepsilon 2} f_2 \rho\varepsilon^2\right)/k$
h_t	$\mu/\Pr + \alpha_t$	0

$$S_U = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\mu_e \frac{\partial U}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_e \frac{\partial V}{\partial x}\right) - \frac{2}{3}\frac{\partial}{\partial x}\left[\frac{\mu_e}{r}\left(\frac{\partial}{\partial x}(rU) + \frac{\partial}{\partial r}(rV)\right) + \rho k\right]$$

$$S_V = -\frac{\partial P}{\partial r} + \frac{\partial}{\partial x}\left(\mu_e \frac{\partial U}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_e \frac{\partial V}{\partial r}\right) - \frac{2}{3}\frac{\partial}{\partial r}\left[\frac{\mu_e}{r}\left(\frac{\partial}{\partial x}(rU) + \frac{\partial}{\partial r}(rV)\right) + \rho k\right]$$

$$G = \mu_t \left\{ 2\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial r}\right)^2 + \left(\frac{V}{r}\right)^2\right] + \left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x}\right)^2 \right\}$$

$$\mu_e = \mu + \mu_t$$

Table 2. Constants in CHC turbulent model [5]

σ_k	σ_ε	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	C_μ
1.0	1.3	1.44	1.92	0.09

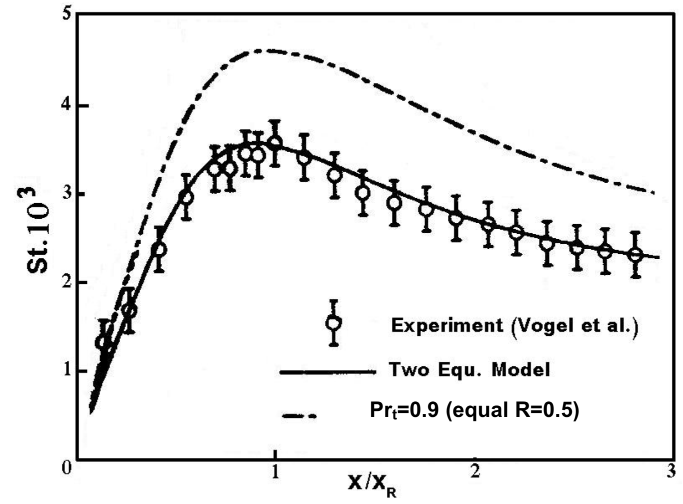

Figure 1. Comparison of calculated Stanton number with two-equation heat transfer model and $Pr_t = 0.9$ [6]

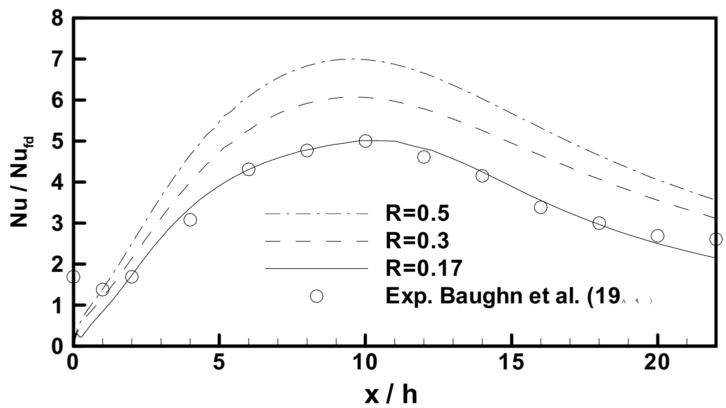

Figure 2. Comparison of the predicted Nusselt number distributions obtained through different R values with the measurements ($Re_D = 17310$ and $d/D = 0.4$)

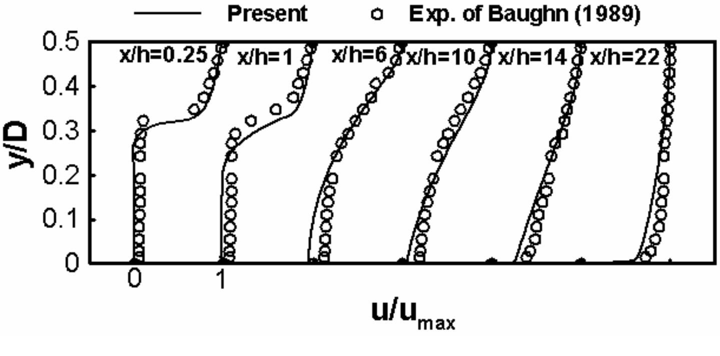

Figure 3. Comparison of computed velocity profiles behind step with experimental measurements ($Re_D = 17310$ and $d/D = 0.4$)

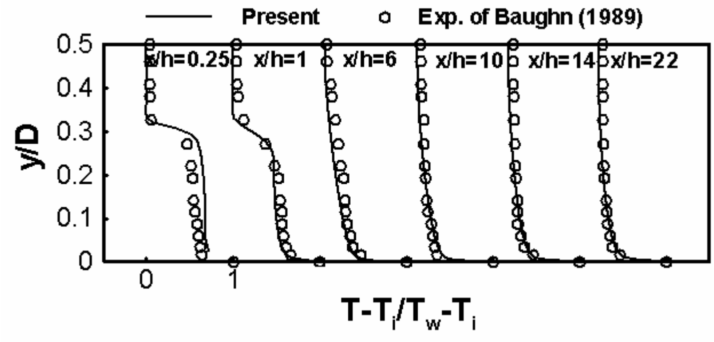

Figure 4. Comparison of computed temperature profiles behind step with experimental measurements (Re_D = 17310 and d/D = 0.4)

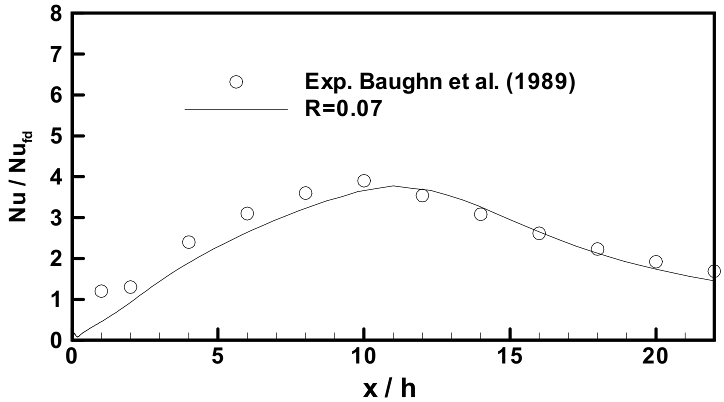

Figure 5. Comparison of the predicted Nusselt number distributions obtained at R=0.07 with the measurements (Re_D = 44540 and d/D = 0.4)

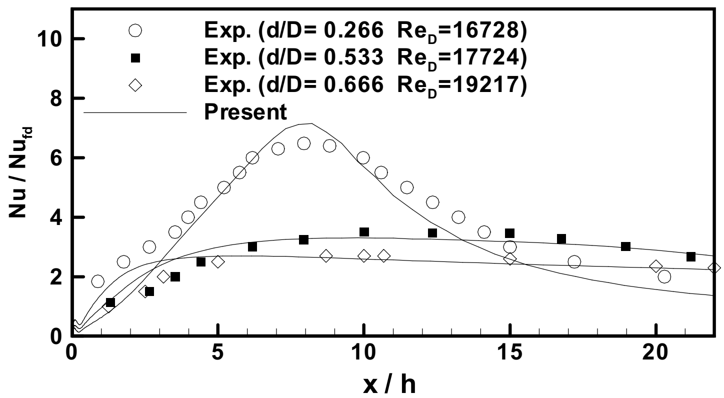

Figure 6. Comparison of the predicted Nusselt number distributions with the measurement [13] at different expansion ratios and Reynolds numbers

Nomenclature

C_μ, C_1, C_2	velocity-field turbulent model constants	u_k	turbulent velocity scale
		u_ε	Kolmogorov velocity scale
D	pipe diameter at outlet	u_τ	friction velocity scale
d	pipe diameter at inlet	V	mean velocity in r-direction
f_μ, f_1, f_2	turbulent model functions	v	fluctuation of V
G	production term in k equation	X_R	reattachment length
h_t	total enthalpy	x, r	co-ordinates in axial and radial directions with x=0 at step location
k	turbulent kinetic energy		
k_t	temperature variance	y	Cartesian coordinate normal to streamwise direction with y=0 at wall
M	molecular weight		
Nu	Nusselt number	y^*, y^{**}	dimensionless distance from wall
P	mean pressure	$\overline{()}$	ensemble-averaged values
Pr	molecular Prandtl number		
Pr_t	turbulent Prandtl number	**Greek symbols**	
R	time scale ratio $(k_t/2\varepsilon_t)/(k/\varepsilon)$	α, α_t	molecular and eddy thermal diffusivities
R_0	universal gas constant	Γ_Φ	transfer coefficient of variable Φ
R_t	local turbulent Reynolds number	ε	dissipation rate of k
Re_d	Reynolds number at inlet	ε_t	dissipation rate of k_t
Re_D	Reynolds number at outlet	ν	kinematic viscosity
S_Φ	source term of variable Φ	ν_t, ν_e	eddy and effective viscosity
St	Stanton number	ρ	density
T	mean temperature	Φ	transport variable
t	fluctuation temperature	σ_k, σ_ε	constants for turbulent diffusion k, ε
U	mean velocity in x-direction		
u	fluctuation of U	**Subscripts**	
		in,i	inlet
		W	wall
		t	turbulent quantity

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